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The stability of a water drop oscillating with finite amplitude in an electric field

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By assuming that an uncharged drop situated in a uniform electric field E retains a spheroidal shape while oscillating about its equilibrium configuration, two approximate equations of motion are derived for the deformation ratio γ expressed as the ratio a/b of the major and minor axis of the drop. Solutions of these equations of motion indicate that the stability of a drop of undistorted radius R and surface tension T depends upon $E(R/T)^{\frac{1}{2}}$ and the initial displacement of γ from its equilibrium value. The predictions of the two equations are compared to assess the accuracy of the spheroidal assumption as applied to such a dynamical situation. The analysis is used to determine the stability criterion of a drop subject to a step function field. Finally, the limit of validity of the spheroidal assumption is discussed in terms of Rayleigh's criterion for the stability of charged spherical drops. By applying Rayleigh's criterion to the poles of a spheroidal drop, the stage at which the drop departs from spheroidal form to form conical jets was approximately determined.

1. Introduction

The behaviour of isolated water drops in electric fields was first studied by Zeleny (1915). He attempted to adopt Rayleigh's (1882) criterion for the stability of a charged drop to the poles of drops in electric fields to study the stability of spheroidal drops in electric fields. Although Zeleny's criterion for the stability of such drops was shown by Taylor (1964) to be incorrect, Zeleny did demonstrate in a later paper (1917) that disintegration of these drops was a result of hydrodynamic instability. Since then, this problem has been studied experimentally by several workers, notably Nolan (1926), Macky (1931) and Ausman & Brook (1967), and the approximate empirical relation $E(R/T)^{\frac{1}{2}} = 1{\cdot}6$ was established between the external field E, expressed in electrostatic units, required to disintegrate a drop of undistorted radius R and surface tension T. Taylor (1964) treated the problem theoretically by assuming that the drop retained a spheroidal shape until it reached the instability point and that the equations of equilibrium between the stresses due to surface tension and electric field were satisfied at the pole and at the equator. He calculated that the onset of instability occurs when $E(R/T)^{\frac{1}{2}} = 1{\cdot}625$ and the deformation γ, expressed as the ratio a/b of the semi-major to semi-minor axes is equal to $1{\cdot}86$. These predictions, which are in good

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agreement with the theoretical results of Brazier-Smith (1971) who calculated the exact shape of uncharged drops in electric fields, are also borne out by the experimental studies of Nolan, Macky and Ausman & Brook and the studies of Wilson & Taylor (1925) on the bursting of soap bubbles in electric fields.

The present study illustrates how the spheroidal assumption, utilized by Taylor to determine the equilibrium of an uncharged drop in an electric field, can be applied to determine the dynamic behaviour of such a drop not in equilibrium. The other assumptions required are that the flow within the drop is irrotational and that viscous damping is negligible. For justification of the last two assumptions, the reader is referred to Lamb (1932), where it is demonstrated that the damping period of the fundamental mode of an oscillating water drop is large compared to its periodic time for drop radii normally encountered.

2. The mathematical approach

The condition of irrotational flow within the drop allows the pressure P to be given by the Bernoulli equation

$$\frac{P}{\rho}+\frac{\partial \phi}{\partial t}+\tfrac{1}{2}(\nabla\phi)^2 = \text{constant}, \tag{1}$$

where ϕ is the velocity potential and ρ the fluid density.

The assumption that the drop remains spheroidal during any deformation is satisfied if the surface is given by the equation

$$\frac{z^2}{a^2}+\frac{r^2}{b^2} = 1, \tag{2}$$

when referred to a cylindrical co-ordinate system (r, z). Hence the undistorted radius R of the drop is given by

$$R^3 = ab^2.$$

The geometry of the system is shown in figure 1. The constraints that the flow is incompressible and the drop remains spheroidal means that the velocity field must necessarily take the form

$$v_r = -A\tfrac{1}{2}r, \quad v_z = Az, \tag{3, 4}$$

where v_r and v_z are the radial and axial velocity components respectively and A is a function of time only. Equations (3) and (4) integrate readily to give the velocity potential

$$\phi = \tfrac{1}{2}A(z^2-\tfrac{1}{2}r^2)+c. \tag{5}$$

It follows from (5) and (1) that the pressure within a drop, in which only incompressible irrotational flow occurs and only spheroidal deformation are permitted, is given by

$$P/\rho = -\tfrac{1}{2}(dA/dt)\,(z^2-\tfrac{1}{2}r^2)-\tfrac{1}{2}A^2(z^2+\tfrac{1}{4}r^2)+c. \tag{6}$$

In fact, (6) is a restricted form of the equation derived by Dirichlet for the pressure inside a liquid ellipsoid oscillating by virtue of its gravitational field; the complete formulation is given by Lamb (1932). The restrictions imposed upon Dirichlet's equation are that the flow is axisymmetric and the gravity potential is absent.

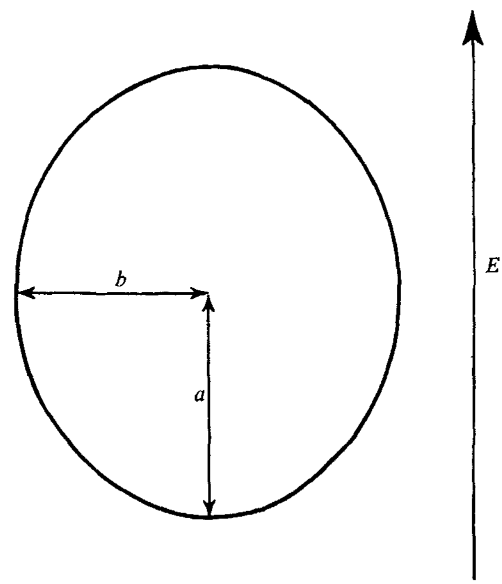

FIGURE 1. The geometry of the system.

3. Derivation of the equation of motion

The polar and equatorial pressures of a prolate spheroidal drop when an electric field E is applied in the direction of its axis was shown by Taylor (1964) to be

$$P_p = 2T\frac{a}{b^2} - \frac{E^2a^2}{8\pi b^2(1-e^2)I^2}, \tag{7}$$

$$P_e = T\left(\frac{b}{a^2}+\frac{1}{b}\right), \tag{8}$$

where $I = 0{\cdot}5e^{-3}\ln\{(1+e)/(1-e)\}-e^{-2}$ and $e = (1-b^2a^{-2})^{\frac{1}{2}}$. $2a$ and $2b$ are, respectively, the distance between the poles of the drop and its equatorial diameter. Substituting (6) into (7) and (8), and eliminating the constant c yields the following equation in A

$$(P_p - P_e)/\rho = -\tfrac{1}{2}(dA/dt)\{a^2\tfrac{1}{2}b^2\} - \tfrac{1}{2}A^2\{a^2-\tfrac{1}{4}b^2\}. \tag{9}$$

At this point it is useful to define the deformation ratio of the drop as $\gamma = a/b$ whence $a = R\gamma^{\frac{2}{3}}$ and $b = R\gamma^{-\frac{1}{3}}$. The condition that (2) shall continue to represent the surface through time requires that

$$\frac{D}{Dt}\left(\frac{z^2}{a^2}+\frac{r^2}{b^2}\right) = 2\left\{-\frac{z^2}{a^3}\frac{da}{dt}-\frac{r^2}{b^3}\frac{db}{dt}+v_z\frac{z}{a^2}+v_r\frac{r}{b^2}\right\} = 0,$$

which, combined with (3) and (4), yields

$$A = \frac{1}{a}\frac{da}{dt} = -\frac{2}{b}\frac{db}{dt}. \tag{10}$$

Substitution of (10) into (9) and rewriting in terms of γ yields equation of motion

$$\frac{d^2\gamma}{dt^2} = \frac{(\frac{1}{3}\gamma^2+\frac{2}{3})}{\gamma(\gamma^2+\frac{1}{2})}\left(\frac{d\gamma}{dt}\right)^2 - \frac{3\gamma(P_p-P_e)}{\rho R^2(\gamma^{\frac{4}{3}}+\frac{1}{2}\gamma^{-\frac{2}{3}})}. \tag{11}$$

This equation may be non-dimensionalized by putting $t' = \omega t$, where ω is the angular frequency of the fundamental mode of an isolated drop as calculated by Rayleigh (1879) and is given by

$$\omega^2 = 8T/R^3\rho.$$

Rewriting (11) in terms of t' and making the appropriate substitution for P_p and P_e from (8) and (9), the equation becomes

$$\frac{d^2\gamma}{dt'^2} = \frac{(\frac{1}{3}\gamma^2+\frac{2}{3})}{\gamma(\gamma^2+\frac{1}{2})}\left(\frac{d\gamma}{dt'}\right)^2 - \frac{3\gamma(2\gamma^2-E_n^2\gamma^{\frac{14}{3}}/8\pi I^2-1/\gamma-\gamma)}{8(\gamma^2+\frac{1}{2})}, \tag{12}$$

where

$$E_n = E(R/T)^{\frac{1}{2}}.$$

Equation (12) is an approximate equation of motion of a drop deforming in an electric field, subject to the constraint that the drop remains spheroidal.

A useful check on the accuracy of (12) for describing the motion of a drop excited in its fundamental mode is to derive another approximate equation of motion using an independent method and compare their solutions. Such a method may use the potential and kinetic energies of a deforming spheroidal drop in an electric field. The expression for potential energy $\mathscr{P}$ of a spheroidal conducting drop immersed in an electrostatic field E has been calculated by O'Konski & Thacher (1953) and is given by

$$\mathscr{P} = 2\pi R^2 T\left(\gamma^{-\frac{2}{3}}+\gamma^{\frac{4}{3}}\frac{\tan^{-1}\epsilon}{\epsilon}\right) - \frac{E_n^2\gamma^2}{6I}R^2T, \tag{13}$$

where $\epsilon = \gamma^2-1$. The kinetic energy $\mathscr{K}$ of a drop oscillating in its spheroidal mode was shown by Billings & Holland (1969) to be

$$\mathscr{K} = \frac{4\pi\rho R^5\gamma^{-\frac{2}{3}}}{135}\left[2+\frac{1}{\gamma^2}\right]\left(\frac{d\gamma}{dt}\right)^2 \tag{14}$$

and since the system is closed, the total energy $\mathscr{T}$ is constant, therefore

$$\mathscr{P}+\mathscr{K} = \mathscr{T} \text{ (constant)}$$

or

$$d\mathscr{P}/dt + d\mathscr{K}/dt = 0. \tag{15}$$

Substituting (13) and (14) into (15) and cancelling throughout by $d\gamma/dt$ yields the second approximate equation of motion:

$$\frac{d^2\gamma}{dt^2} = \frac{(\frac{1}{3}\gamma^2+\frac{2}{3})}{\gamma(\gamma^2+\frac{1}{2})}\left(\frac{d\gamma}{dt}\right)^2 - \frac{135\, d\mathscr{P}/d\gamma}{8\pi\rho R^5(2\gamma^{-\frac{2}{3}}+\gamma^{-\frac{8}{3}})}, \tag{16}$$

which, of course, may be put in dimensionless form as was (11). We note that the inertial terms [i.e. the coefficients of $(d\gamma/dt)^2$] in (11) and (16) are identical and that the only essential difference between the two equations is the forcing terms.

4. Integration of the equations of motion

Since the analytical integration of the two equations of motion would have been extremely tedious, numerical methods were employed for their solution. The equations of motion yield their solutions readily by specifying γ and $d\gamma/dt'$ at time $t' = 0$ and application of a marching process, or one-step method. If γ_0 and v_0 are the values assigned to γ and $d\gamma/dt'$ at zero time then the marching process adopted may be described by the following equations which apply to the nth step:

$$\gamma_{n+1} = \gamma_n + v_n\Delta + \tfrac{1}{2}(a_n + \tfrac{1}{3}g_n)\Delta^2, \tag{17}$$

$$v_{n+1} = v_n + (a_n + \tfrac{1}{2}g_n)\Delta, \tag{18}$$

where Δ = time difference between each step,

$$v_n = d\gamma/dt' \text{ at the } n\text{th step},$$
$$a_n = d^2\gamma/dt'^2 \text{ at the } n\text{th step},$$
$$g_n = a_n - a_{n-1} \simeq (d^3\gamma/dt'^3)\Delta.$$

Equations (17) and (18) correspond to the first few terms of a Taylor series, namely

$$\gamma(t'+\Delta) \simeq \gamma(t') + \frac{d\gamma}{dt'}\Delta + \frac{1}{2}\frac{d^2\gamma}{dt'^2}\Delta^2 + \frac{1}{6}\frac{d^3\gamma}{dt'^3}\Delta^3$$

and

$$\frac{d\gamma}{dt'}(t'+\Delta) \simeq \frac{d\gamma}{dt'}(t) + \frac{d^2\gamma}{dt'^2}\Delta + \frac{1}{2}\frac{d^3\gamma}{dt'^3}\Delta^2.$$

At each step γ_n and v_n are given by the previous step and a_n is derived analytically from the appropriate equation of motion.

When the above scheme was applied to solve the simple harmonic equation, which is similar to our equations of motion, it was found to provide solutions that were accurate to 0·01 % for a time increment of 1/100 the period.

If we assume that the period of the drop in an electric field is of the same order as the period τ of an isolated drop then Δ may be taken to be a suitable fraction of τ. In terms of the parameter t' the period of an isolated drop is 2π and, on this basis, the value of Δ was chosen to be $0{\cdot}001\,\pi$ or $\tau/2000$ which is much smaller than the time period used for solving the simple harmonic equation.

5. Solutions

The solutions of (11) and (16) may be determined completely by two dimensionless parameters. For the first parameter we have a choice between three dependent quantities, namely, the maximum value of γ, $\gamma_{\max}$, the minimum value of γ, $\gamma_{\min}$ and finally $\gamma_d = \gamma_{\max} - \gamma_{\min}$. The second parameter may be either the equilibrium deformation γ_e or the dimensionless field E_n. The predicted variation of γ_e with E_n will depend upon which equation of motion we adopt; if (11) is adopted then the variation of γ_e is as computed by Taylor (1964) and if (16) is adopted then the variation is as computed by Rosenkilde (1969). Both results are illustrated in figure 2 and can be seen to correspond very closely. Also shown on the figure is the variation of the deformation required to produce a zero pressure at the poles of the drop (Zeleny's criterion) with applied field. The reason for its inclusion is discussed later.

Figure 3 shows three solutions of (12) for $\gamma_{\min} = 1{\cdot}0$, $1{\cdot}1$ and $1{\cdot}2$. In all three cases $\gamma_e = 1{\cdot}5$, $E_n = 1{\cdot}559$. It is apparent from the figure that γ_d has a critical value for which the solutions undergo a transition from periodic to asymptotic form.

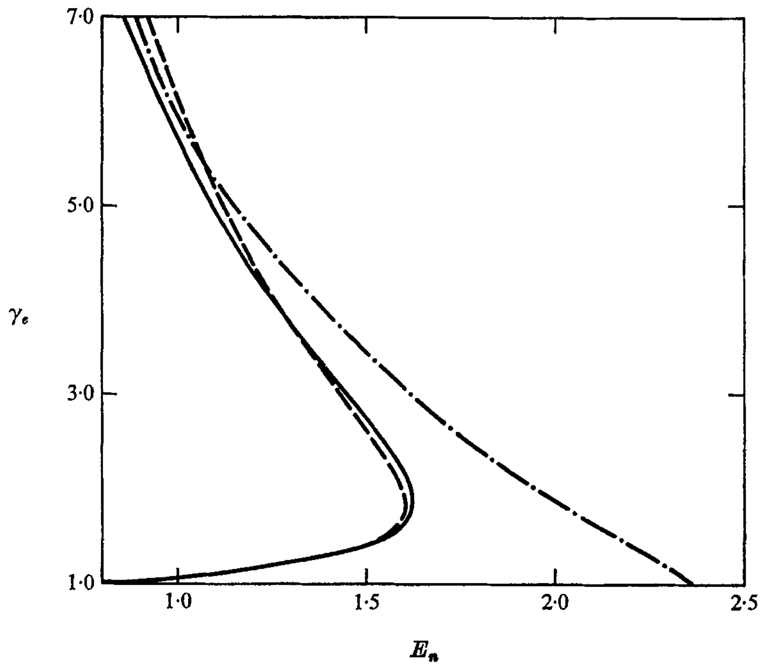

FIGURE 2. The variation of γ_e with E_n. Solid line represents approximation I; dashed line represents approximation II and the third line is Zeleny's criterion.

To obtain a better physical picture of the critical transition of the solution from periodic to asymptotic, one more parameter, the non-dimensional frequency f_n, of the drop is required. Its defining equation is

$$f_n = \pi f (R^3 \rho / 2T)^{\frac{1}{2}},$$

where f is the dimensional frequency of the drop. The dimensionless frequency for a drop oscillating with small amplitude in the absence of any electric field will therefore be $f_n = 1$. By generating more solutions to (12) and (16), two approximate values of f_n may be found for various values of γ_d and γ_e. The variation of f_n with γ_d, as computed from (12) and (16) for various values of γ_e, is illustrated in figure 4. We shall refer subsequently to (11) and (16) and the related solutions as approximations I and II. The figure illustrates that there is good agreement between the two frequencies as predicted by approximations I and II for small amplitudes of oscillation. As γ_d increases, however, the disparity between the two approximations increases. For, $\gamma_d = 1$ the increase in error is about $0{\cdot}05 f_0$, where f_0 is the fundamental frequency as calculated by Rayleigh (1879). The

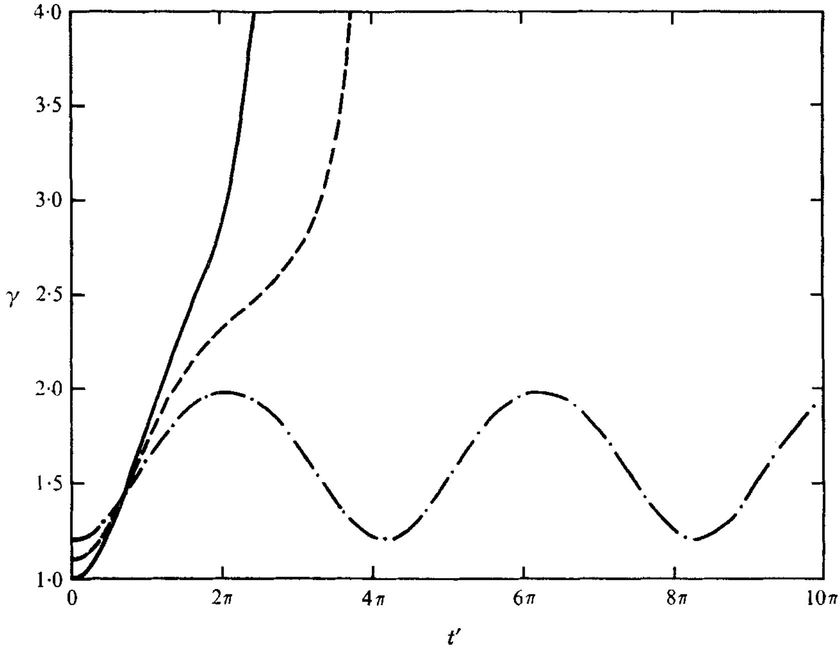

FIGURE 3. The variation of deformation γ of a drop with time t' according to approximation I, for γ_{min} = 1·0, 1·1 and 1·2. In all three cases, γ_e = 1·5.

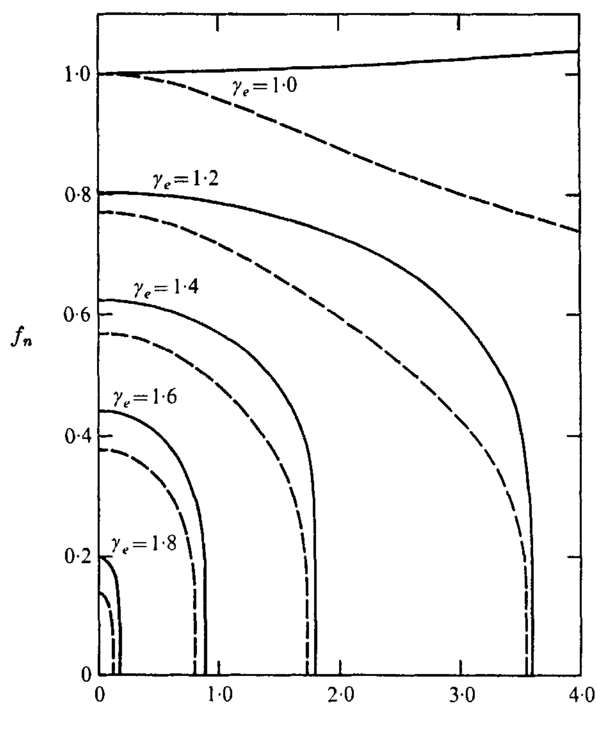

FIGURE 4. The dimensionless frequency f_n as a function of γ_d for various values of γ_e. Solid line represents approximation I; dashed line represents approximation II.

degree of agreement between the two approximations for γ_d tending to zero is better illustrated in figures 5 and 6 which show the variation of f_n with γ_e and E_n respectively for the limit $\gamma_d = 0$. For this limit, the frequency f_n for approximation I tends to the frequency of small amplitude oscillations as calculated by Brazier-Smith *et al.* (1971); similarly approximation II tends to the frequency as predicted by Rosenkilde (1969) who used a method described by Chandrasekhar (1961, 1965, 1969) for the analysis of fluid problems.

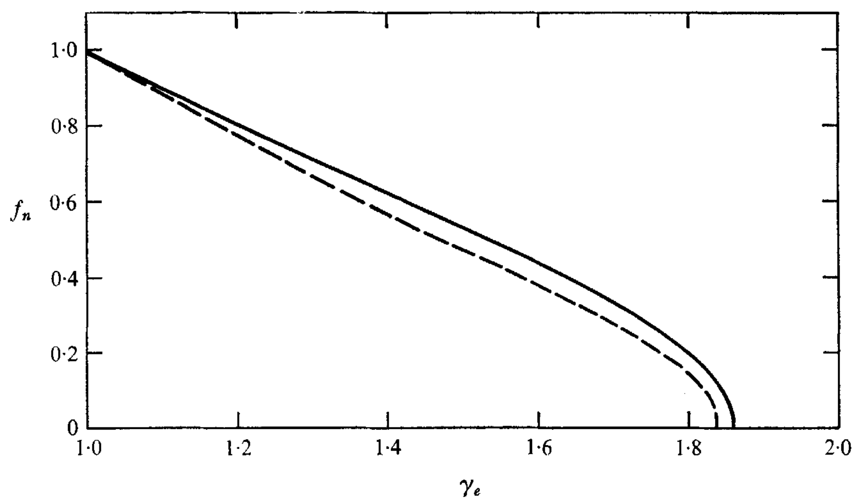

FIGURE 5. The variation of f_n for small amplitudes with γ_e. Solid line represents approximation I; dashed line represents approximation II.

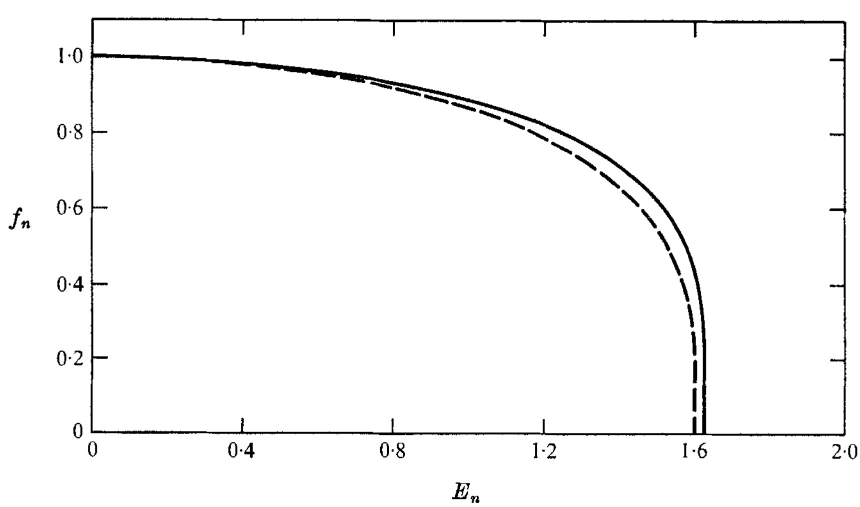

FIGURE 6. The variation of f_n for small amplitudes with E_n. Solid line represents approximation I; dashed line represents approximation II.

6. Instability criteria

In the last section, we observed that solutions of (12) and (16) are either periodic or asymptotic depending upon γ_d and E_n, thus determining whether the drop is stable. Figure 4 illustrates that the transition from stable to unstable state occurs when f_n becomes zero. However, there is another way of looking at this instability. Taylor (1964) showed that as illustrated in figure 2, there are two equilibrium deformations γ_e for a drop situated in an electric field E_n. One is stable, in which $\partial\gamma_e/\partial E_n$ is positive and the other is unstable. If a drop possesses a

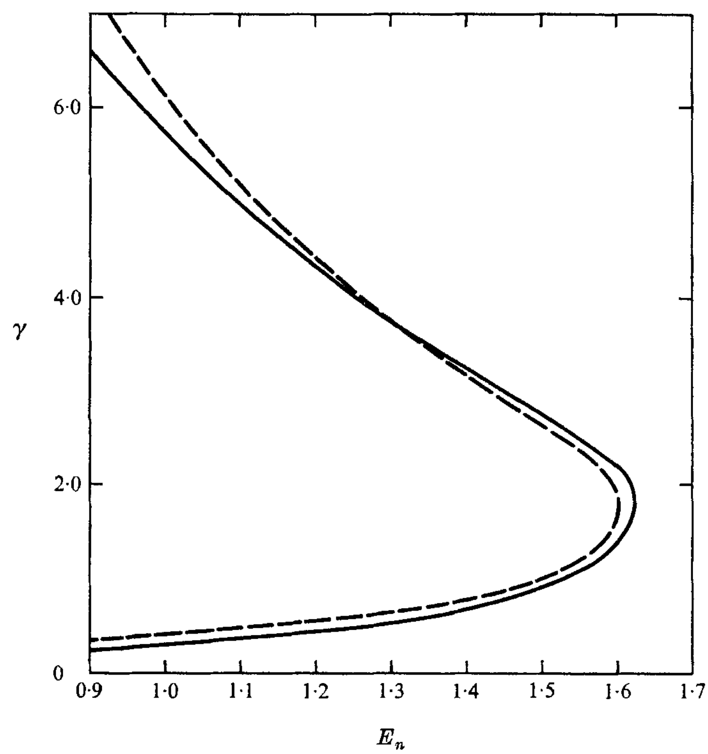

FIGURE 7. The variation of critical values of γ_{max} and γ_{min} against E_n. Solid line represents approximation I; dashed line represents approximation II.

deformation such that γ_e(unstable) $> \gamma > \gamma_e$(stable) and $d\gamma/dt = 0$ then the drop will move back towards the stable equilibrium. The condition that a drop should become unstable is that it has sufficient kinetic energy in its $n = 2$ mode to reach the unstable equilibrium point. This is well illustrated in figure 7 which shows the critical maximum and minimum values of γ for approximations I and II for the drop to be just stable. It can be seen that the critical maximum values of γ for approximations I and II correspond exactly with the two approximate unstable equilibrium values shown in figure 2.

7. Behaviour of drops in step function fields

Since the equilibrium shape of a drop in an electric field is very nearly spheroidal for *all* stable shapes, the effect of suddenly changing, or applying an electric field to a drop in equilibrium will be primarily to excite the $n = 2$ mode; the relative excitation of higher modes will be very small. The importance of considering this problem arises because of two reasons. (i) The sudden rearrangement of charge accompanying a lightning discharge will cause sudden changes in the electric field. This in turn will cause drops to vibrate in their spheroidal mode and it has been shown by Brook & Latham (1968) that it is in principle possible to determine the frequency and magnitude of these vibrations by analysing the return signal

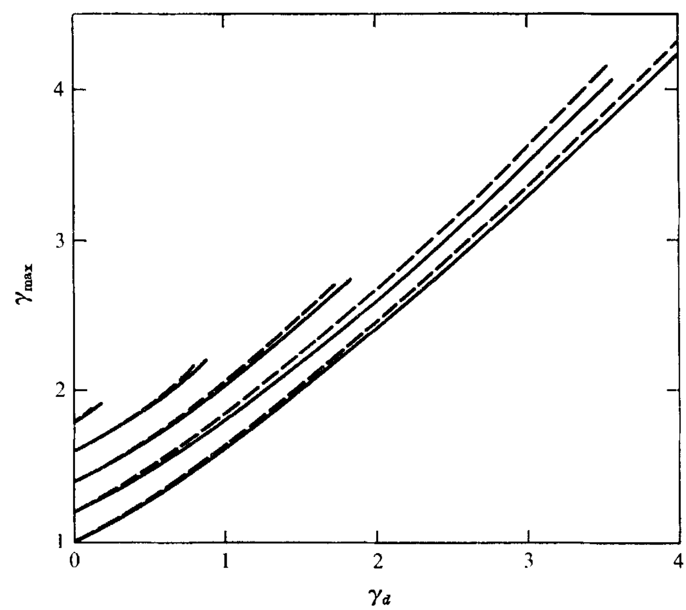

FIGURE 8. The variation of γ_{max} with γ_d for various values of γ_e. Solid line represents approximation I; dashed line represents approximation II.

from a radar beam. Thus, it may be possible to obtain considerable information concerning the mechanism of charge drainage and the lightning stroke itself. (ii) A number of experimental studies have been carried out (notably Macky 1931) to establish the criteria for stability of drops in electric fields that involve the sudden application of an electric field. If their results are to be reconciled with the quasi-static instability criterion then the inertial effect will, as pointed out by Ausman & Brook (1967), have to be taken into account.

In order to determine the frequency of oscillation of a drop when the applied field is suddenly changed, it is necessary to find the maximum and minimum values of deformation, γ_{max} and γ_{min} for a drop oscillating with finite amplitude γ_d about a given equilibrium deformation γ_e. This may be done by referring to figure 8 which shows the variation of γ_{max} with γ_d for values of γ_e. We now note that the electric field before the change is that field required to produce an equilibrium deformation of either $\gamma_{\min}$ or $\gamma_{\max}$ depending upon whether the field is stepped up or down: the field after the change is then that field required to produce an equilibrium deformation of γ_e. The oscillation frequency of a drop may then be determined by finding $\gamma_{\max}$ or $\gamma_{\min}$, whichever is appropriate from figure 8, in conjunction with figure 2: the frequency can then be read directly from figure 4 for the selected values of γ_e.

If the electric field is stepped down then the only effect on the drop present will be to cause it to oscillate. However, if a field is stepped up, it is possible that the drop may break up depending upon whether $\gamma_{\min}$ (the equilibrium deformation before the increase in field) is less than the critical minimum value of γ as illustrated in figure 7. On this basis, we can calculate the critical condition that a

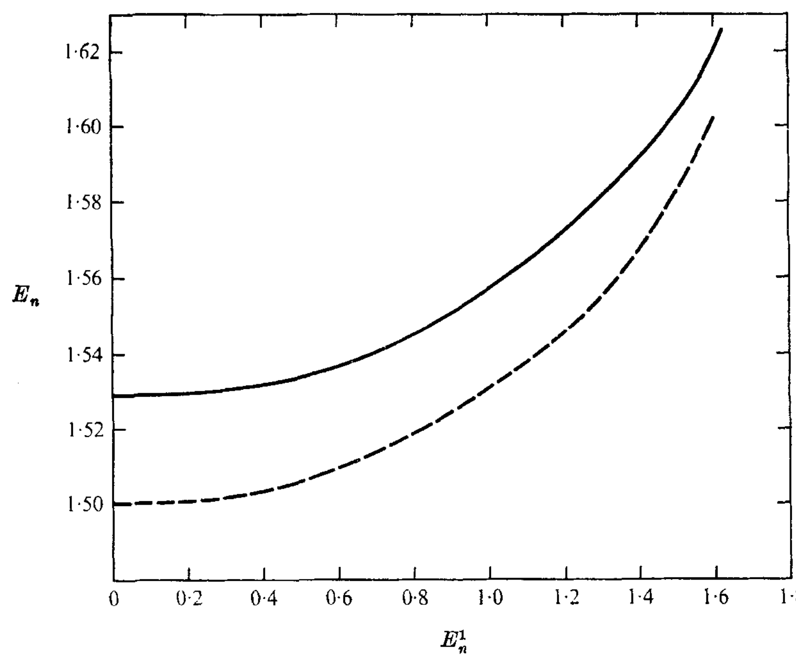

FIGURE 9. The variation of the critical field E_n applied impulsively to disrupt a drop in a field E_n'. Solid line represents approximation I; dashed line represents approximation II

dimensionless field E_n' will disrupt a drop when it is suddenly increased to a value of E_n. This critical condition is illustrated in figure 9 which shows the variation of the critical value of E_n with E_n' for approximations I and II. Again, the indication is that the degree of correspondence of the two approximations is good. One result is that the critical value of E_n required to break up a drop if it is suddenly applied to a drop initially in zero electric field lies between 1·50 and 1·53. This is especially interesting since Macky's (1931) experimental work on the break up of drops as they fall between parallel capacitor plates, resulting in such a suddenly applied electric field, predicted that $E_n = 1{\cdot}51$ for a drop to break up, which lies between the two values determined from approximations I and II. This close correspondence of Macky's value of E_n and the two values determined above provides evidence for the preceding analysis.

8. Instability and excitation in the higher modes

The onset of instability in modes higher than the spheroidal mode has been observed by Zeleny (1917), Macky (1931) and others to occur soon after the onset of instability in the spheroidal mode. Although Taylor (1964) showed theoretically that the final result of higher modal instability is the formation of conical ends to the drop, no work has been done to determine satisfactorily at what point the higher modal instability is initiated, which must be known in order to delimit the acceptable range in which (12) and (16) are valid. While it is desirable to determine this point exactly, we shall, because of the complexity of the problem, have to rely in some degree upon qualitative arguments and scanty experimental results to establish approximately where this point lies.

Rosenkilde (1969) puts an upper limit on γ, for the spheroidal assumption to be valid, when $\int P\,dV$ over the volume of the drop becomes zero. Using this criterion for a drop immersed in an electric field of strength given by

$$E(R/T)^{\frac{1}{2}} = 1{\cdot}6,$$

the spheroidal assumption becomes invalid at $\gamma = 4{\cdot}5$. Since no cases have been recorded where spheroidal conducting drops attain such deformations, we can conclude that higher modal instability sets in before Rosenkilde's limit is reached.

To obtain some idea of the criterion for the onset of instability of higher modes, we shall invoke Rayleigh's (1882) criterion for the stability of a spherical conducting drop carrying an electric charge. In terms of its radius R, surface tension T and the field E on the surface induced by the charge, this criterion is

$$E^2/16\pi < T/R.$$

If this inequality is not satisfied, then Rayleigh's analysis predicts that at least the spheroidal mode, whose wavelength is πR, is unstable. By applying this criterion to the pole of a spheroidal drop, where the radius of curvature is b^2/a, we obtain the following criterion

$$E^2a^2/8\pi b^2(1-e^2)I < 2Ta/b^2. \tag{19}$$

All the symbols correspond to those used in (8). If this inequality is not satisfied then, in so far as Rayleigh's criterion can be carried over to this case, there will be an unstable mode of wavelength $\pi b^2/a$ which is shorter than the wavelength of the spheroidal mode; therefore a higher mode has become unstable and the spheroidal assumption is invalid. The above criterion is, of course, the same criterion that Zeleny (1915) mistakenly used for the stability of spheroidal mode and is illustrated in figure 2.

The flaw in the above argument lies in the fact that Rayleigh's criterion applies over the entire surface of a spherical drop while the criterion expressed by (19) applies only at the poles. However, some experimental data has been obtained which will help us to establish where higher model instability sets in. Macky (1931) reports that the drops used in his experiments achieved deformations of γ between 3 and 4 before the drop exhibited high mode instability. In fact, his photographs indicate that the figure is probably nearer 3 than 4. Since his drops became unstable at field strength of $E_n = 1{\cdot}51$, we have established one point for the instability which we shall take as $\gamma = 3$, $E_n = 1{\cdot}51$. In their experiments, Ausman & Brook (1967) report that $\gamma = 2{\cdot}2$ when high mode instability occurs and their value of E_n was $1{\cdot}56 \pm 0{\cdot}1$. These two results suggest that the instability of the higher modes is initiated somewhere between Taylor's unstable equilibrium point and Zeleny's criterion.

It may be possible to resolve the problem of onset of instability of the higher modes by the following consideration. The general solution for the velocity potential within a prolate spheroidal boundary is, as given by Lamb (1932),

$$\phi = P_n(\mu)P_n(\zeta). \tag{20}$$

μ and ζ are ellipsoidal co-ordinates and satisfy the following:

$$z = k\mu\zeta \quad \text{and} \quad r = k(1-\mu^2)^{\frac{1}{2}}(\zeta^2-1)^{\frac{1}{2}},$$

where $k = (a^2-b^2)^{\frac{1}{2}}$.

The surface of the drop is defined by $\zeta = a/k$. The special case of $n = 2$, giving (5), is that dealt with in the present paper. Velocity potentials of the form given by (20) for $n > 2$ correspond to higher modal excitation and their analysis may yield information about the stability of the higher modes.

The problem of a drop oscillating in its fundamental mode is, of course, a non-linear one and one would intuitively expect that, for spheroidal oscillations of *finite* amplitude, there will be some coupling between the spheroidal and higher modes. This has been borne out by some studies by G. Brant Foote (private communication) on finite amplitude oscillations of drops under the action of surface tension. In fact, the divergence of the two approximations with increasing amplitude as shown in figure 4 is indicative of the increasing coupling with amplitude, the effect of which will be the removal of energy from the $n = 2$ mode to the higher modes where viscous damping will be more efficient. Therefore the large amplitude oscillations will be more difficult to achieve and maintain; in fact the largest oscillations recorded are by Jones (1959) who reported values of $0{\cdot}9$ for γ_d. Therefore the present calculations cannot be considered realistic beyond this point.

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